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Research Article

The Polynomial Sequence Generalizing the Integer Sequence which Enumerates the Number of Subsets of the Set [n] Including No Two Consecutive Even Integers

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Abstract

Fibonacci polynomial sequence is an extension of Fibonacci sequence. Here we define a polynomial sequence generalizing the integer sequence which enumerates the number of subsets of the set [n] including no two consecutive even integers. The polynomial sequence is associated with the Fibonacci polynomials. Some basic properties of the polynomial sequence are obtained.

Keywords: Fibonacci numbers, Fibonacci polynomials, polynomial sequence, consecutive even integers, generating function, combinatorial representation.

[n] Kümesinin Ardışık İki Çift Tamsayı İçermeyen Alt Kümelerinin Sayısını Veren Tamsayı Dizisini Genelleyen Polinom Dizisi

Öz

Fibonacci polinom dizisi Fibonacci dizisinin bir genişlemesidir. Burada [n] kümesinin ardışık iki tamsayı içermeyen alt kümelerinin sayısını veren tamsayı dizisini genelleyen bir polinom dizisi tanımladık. Bu polinom dizisi Fibonacci polinomları ile ilişkilendirildi. Polinom dizisinin bazı temel özellikleri elde edildi.

Anahtar Kelimeler: Fibonacci sayıları, Fibonacci polinomları, Poinom dizisi, Ardışık çift sayılar, Üreteç fonksiyon, Kombinatoryal gösterim.

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1. Introduction

The Fibonacci polynomials are a polynomial sequence which can be considered as a generalization of the Fibonacci numbers. You can see more about Fibonacci polynomials in [5]. The polynomials $F_n(x)$ studied by Belgian Mathematician Eugene Charles Catalan are defined by the recurrence relation as follows:

$$F_n(x) = \begin{cases} 0, & if\ n = 0 \\ 1, & if\ n = 1 \\ xF_{n-1}(x) + F_{n-2}(x), & if\ n \geq 2 \end{cases} \quad (1)$$

When $x = 1$ and $x = 2$, we obtain respectively the nth Fibonacci number F_n and the nth Pell number P_n. Generating function for Fibonacci polynomial sequence and Binet's formula of Fibonacci polynomials are given in [5] by

$$\sum_{n=0}^{\infty} F_n(x)\, t^n = \frac{t}{1 - xt - t^2}, \quad (2)$$

$$F_n(x) = \frac{(x + \sqrt{x^2+4})^n - (x - \sqrt{x^2+4})^n}{2^n\sqrt{x^2+4}}. \quad (3)$$

Consider the sequence $(a_n)_{n\geq 0}$ which enumerates the number of subsets S of the set [n] = $\{1, 2, \ldots, n\}$ such that S contains no two consecutive odd integers. You can see [6] for recursive definition, the generating function, the closed form formula and the sum of first n terms of the sequence $(a_n)_{n\geq 0}$:

$$a_n = 2a_{n-2} + 4\,a_{n-4}, \quad n > 3, \quad (4)$$

$$a_0 = 1,\ a_1 = 2,\ a_2 = 4,\ a_3 = 8.$$

$$F(x) = \sum_{k=0}^{\infty} a_k\, x^k = \frac{4x^3 + 2x^2 + 2x + 1}{1 - 2x^2 - 4x^4}. \quad (5)$$

$$a_{2n} = 2^n F_{n+2}. \quad (6)$$

$$a_{2n+1} = 2^{n+1} F_{n+2}. \quad (7)$$

$$\sum_{0\leq k\leq 2n} a_k = \frac{2^{n+1}}{5}[F_{n+3} + F_{n+5}] - \frac{9}{5}. \quad (8)$$

$$\sum_{0\leq k\leq 2n+1} a_k = \frac{3.2^{n+1}}{5}[F_{n+2} + F_{n+4}] - \frac{9}{5}. \quad (9)$$

In this paper we first define the polynomial sequence $(a_n(\mathrm{x}))_{n\geq 0}$ using (4) and we obtain some basic properties of the polynomial sequence.

2. Main Results

2.1. Recursive definition of the polynomial sequence

Let's define the polinomial sequence $(a_n(x))$ like Eugene Charles Catalan defined in (1) for $F_n(x)$:

$$a_n(x) = \begin{cases} 1, & if\ n = 0 \\ 2, & if\ n = 1 \\ 4, & if\ n = 2 \\ 8, & if\ n = 3 \\ 2x^2 a_{n-2}(x) + 4a_{n-4}(x), & if\ n \geq 4 \end{cases} \quad (10)$$

The first few polynomials are:

$$a_0(x) = 1$$

$$a_1(x) = 2$$

$$a_2(x) = 4$$

$$a_3(x) = 8$$

$$a_4(x) = 8x^2 + 4$$

$$a_5(x) = 16x^2 + 8$$

$$a_6(x) = 16x^4 + 8x^2 + 16$$

$$a_7(x) = 32x^4 + 16x^2 + 32$$

$$a_8(x) = 32x^6 + 16x^4 + 64x^2 + 16$$

Notice that $a_n(1) = a_n$ which is A279312 in the On-Line Encyclopedia of Integer Sequences (OEIS) [2].

2.2. Generating function and the closed form formula of the polynomial sequence $(a_n(x))$

Let's try to find generating function $G(x,t)$ of the polynomial sequence $(a_n(x))$ using the formal power series.

$$G(x,t) = \sum_{n=0}^{\infty} a_n(x) t^n$$

To find $G(x,t)$, multiply both sides of the recurrence relation (10) by t^n and sum over the values of n for which the recurrence is valid, namely, over $n \geq 4$. We get,

$$\sum_{n\geq 4} a_n(x)t^n = \sum_{n\geq 4} 2x^2 a_{n-2}(x)t^n + \sum_{n\geq 4} 4a_{n-4}(x)t^n. \quad (11)$$

Then try to relate these sums to the unknown generating function $G(x,t)$. We have

$$\sum_{n\geq 4} a_n(x)t^n = G(x,t) - a_0(x) - a_1(x)t - a_2(x)t^2 - a_3(x)t^3$$

$$= G(x,t) - 1 - 2t - 4t^2 - 8t^3,$$

$$\sum_{n\geq 4} 2x^2 a_{n-2}(x)t^n = 2x^2t^2 \sum_{n\geq 4} a_{n-2}(x)t^{n-2}$$

$$= 2x^2t^2\,(G(x,t) - a_0(x) - a_1(x)t)$$

$$= 2x^2t^2\,(G(x,t) - 1 - 2t),$$

$$\sum_{n\geq 4} 4a_{n-4}(x)t^n = 4t^4 \sum_{n\geq 4} 4a_{n-4}(x)t^{n-4}$$

$$= 4t^4 G(x,t).$$

If we write these results on the two sides of (11), we find that

$$G(x,t) - 1 - 2t - 4t^2 - 8t^3$$
$$= 2x^2t^2\,(G(x,t) - 1 - 2t) + 4t^4 G(x,t)$$

which is trivial to solve for the unknown generating function $G(x,t)$, in the form

$$G(x,t) = \frac{1 + 2t + (4 - 2x^2)t^2 + (8 - 4x^2)t^3}{1 - 2x^2t^2 - 4t^4}. \quad (12)$$

Substituting $x = 1$, we get the generating function for the sequence $(a_n)_{n\geq 0}$ which is given in (5).

Theorem 1. *Let $(a_n(x))$ is the polynomial sequence defined by (10). Then we have*

$$a_{2n}(x) = 2^n[F_{n-1}(x^2) + 2F_n(x^2)],$$

$$a_{2n+1}(x) = 2^{n+1}[F_{n-1}(x^2) + 2F_n(x^2)],$$

where $F_n(x)$ is the nth Fibonacci polynomial with the Binet's formula

$$F_n(x) = \frac{(x + \sqrt{x^2+4})^n - (x - \sqrt{x^2+4})^n}{2^n\sqrt{x^2+4}}.$$

Proof. If $A(x,t)$ is the generating function for even terms of the polinomial sequence $(a_n(x))$ then it is clear that $A(x,t) = \frac{1}{2}(G(x,t) + G(x,-t))$. Using generating function (12) we get,

$$A(x,t) = \frac{1 + (4 - 2x^2)t^2}{1 - 2x^2t^2 - (2t^2)^2}.$$

Substituting u for $2t^2$we have

$$A(x,u) = \frac{1 + (2 - x^2)u}{1 - x^2u - u^2}$$

$$= \frac{1}{1 - x^2u - u^2} + (2 - x^2)\frac{u}{1 - x^2u - u^2} \quad (13)$$

The generating function of the Fibonacci polynomial given by (2) is

$$f(x,t) = \frac{t}{1 - xt - t^2}$$

$$= 0t^0 + 1t^1 + xt^2 + (x^2+1)t^3 + \ldots + F_n(x)t^n + \ \ldots \quad (14)$$

Let's indicate the correspondence between a sequence and its generating function with a double-sided arrow as follows:

$$\langle 0, 1, x, x^2+1, \ldots \rangle \quad \leftrightarrow \quad \frac{t}{1 - xt - t^2} \quad (15)$$

$$\langle 1, x, x^2+1, \ldots \rangle \quad \leftrightarrow \quad \frac{1}{1 - xt - t^2} \quad (16)$$

If we right-shift the polynomial sequence in (16) by adding one leading zero, we obtain the polynomial sequence in (15). Hence the generating function of the polynomial sequence $F_{n+1}(x)$ is

$$\frac{1}{1 - xt - t^2} =$$

$$1t^0 + xt^1 + (x^2+1)t^2 + \ldots + F_{n+1}(x)t^n + \ \ldots \quad (17)$$

Substituting x^2 for x into the equation (17) and replacing t with u we have

$$\frac{1}{1 - x^2u - u^2} =$$

$$1u^0 + x^2u^1 + (x^4+1)u^2 + \ldots + F_{n+1}(x^2)u^n + \ \ldots$$

Substituting $u = 2t^2$ in the right side of the equation we have

$$\frac{1}{1 - x^2u - u^2} = 1(2t^2)^0 + x^2(2t^2)^1$$

$$+(x^4+1)(2t^2)^2 + \ldots + F_{n+1}(x^2)(2t^2)^n + \ \ldots$$

$$= 1.2^0t^0 + x^2 2^1 t^2 + (x^4+1).2^2t^4 + \ldots$$

$$+F_{n+1}(x^2).2^n t^{2n} + \ \ldots \quad (18)$$

Using (15) we have

$$\frac{u}{1 - x^2u - u^2} =$$

$$0u^0 + 1u^1 + x^2u^2 + (x^4+1)u^3 + \ldots + F_n(x^2)u^n + \ldots$$

Substituting $u = 2t^2$ in the right side of the equation we have

$$\frac{u}{1 - x^2u - u^2} =$$

$$1(2t^2)^1 + x^2(2t^2)^2 + (x^4+1)(2t^2)^3 + \ldots$$

$$+F_n(x^2)(2t^2)^n+\ ...$$

$$= 1.2^1t^2 + x^2 2^2 t^4 + (x^4+1).2^3t^6 +$$

$$...+F_n(x^2).2^n t^{2n} +\ ... \quad (19)$$

Substituting (18) and (19) into the equation (13) we get the coefficients of t^{2n} which gives the general term for the polynomial subsequence $(a_{2n}(x))$

$$a_{2n}(x) = 2^n[F_{n+1}(x^2) + (2-x^2)F_n(x^2)]. \quad (20)$$

Since $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ we have $F_{n+1}(x^2) = x^2F_n(x^2) + F_{n-1}(x^2)$. Using this fact and (20) we obtain

$$a_{2n}(x) = 2^n[F_{n-1}(x^2) + 2F_n(x^2)]. \quad (21)$$

If $B(x,t)$ is the generating function for odd terms of the polinomial sequence $(a_n(x))$ then it is clear that $B(x,t) = \frac{1}{2}(G(x,t) - G(x,-t))$. Using generating function (12) we get,

$$B(x,t) = \frac{2t + (8-4x^2)t^3}{1-2x^2t^2-(2t^2)^2}$$

$$= t\frac{2}{1-2x^2t^2-(2t^2)^2} + t(4-2x^2)\frac{2t^2}{1-2x^2t^2-(2t^2)^2})$$

Using (18) we have

$$\frac{2}{1-2x^2t^2-(2t^2)^2}$$

$$= 1.2^1t^0 + x^2 2^2t^2 + (x^4+1).2^3t^4 + ...$$

$$+F_{n+1}(x^2).2^{n+1}t^{2n} +\ ...$$

$$t\frac{2}{1-2x^2t^2-(2t^2)^2}$$

$$= 1.2^1t^1 + x^2 2^2t^3 + (x^4+1).2^3t^5 + ...$$

$$+F_{n+1}(x^2).2^{n+1}t^{2n+1} + ... \quad (22)$$

Using (19) we have

$$t\frac{2t^2}{1-2x^2t^2-(2t^2)^2}$$

$$= 1.2^1t^3 + x^2 2^2t^5 + (x^4+1).2^3t^7 + ...$$

$$+F_n(x^2).2^n t^{2n+1} +\ ... \quad (23)$$

From (22) and (23) we get the coefficients of t^{2n+1} which gives the general term for the polynomial subsequence $(a_{2n+1}(x))$

$$a_{2n+1}(x) = 2^{n+1}[F_{n+1}(x^2) + (2-x^2)F_n(x^2)] \quad (24)$$

Since $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, we have $F_{n+1}(x^2) = x^2F_n(x^2) + F_{n-1}(x^2)$. Using this fact and (24) we obtain

$$a_{2n+1}(x) = 2^{n+1}[F_{n-1}(x^2) + 2F_n(x^2)], \quad (25)$$

where

$$F_n(x) = \frac{(x+\sqrt{x^2+4})^n - (x-\sqrt{x^2+4})^n}{2^n\sqrt{x^2+4}}.$$

The proof is completed.

Notice that $a_{2n}(1) = a_{2n} = 2^nF_{n+2}$ which is given in (6).

Notice that $a_{2n+1}(1) = a_{2n+1} = 2^{n+1}F_{n+2}$ which is given in (7).

It is clear that (21) and (25) implies

$$a_{2n+1}(x) = 2\,a_{2n}(x)$$

2.3. The sum of the first n terms of the polynomial sequence

The sum of the first n Fibonacci polynomials is given in [3] as follows:

$$\sum_{i=1}^{n} F_i\,(x) = \frac{F_{n+1}(x) + F_n(x) - 1}{x}.$$

Theorem 2. *Let $(a_n(x))$ is the polynomial sequence defined by (10). Then we have*

$$\sum_{k=0}^{2n} a_k\,(x)$$

$$= \frac{2^{n+1}[(2x^2+9)F_n(x^2) + (9-2x^2)F_{n-1}(x^2)] + 6x^2 - 15}{2x^2+3},$$

$$\sum_{k=0}^{2n+1} a_k\,(x)$$

$$= \frac{2^{n+1}[(6x^2+15)F_n(x^2) + 12F_{n-1}(x^2)] + 6x^2 - 15}{2x^2+3},$$

where $F_n(x)$ is the nth Fibonacci polynomial with the Binet's formula

$$F_n(x) = \frac{(x+\sqrt{x^2+4})^n - (x-\sqrt{x^2+4})^n}{2^n\sqrt{x^2+4}}.$$

Proof. Let $(S_n(x))_{n\geq 0}$ be the sum of first n terms of the polynomial sequence $(a_n(x))$:

$$S_n(x) = \sum_{k=0}^{n} a_k\,(x)$$

Using recurrence relation (10) and its initial conditions we have

$$a_n(x) = 2x^2a_{n-2}(x) + 4a_{n-4}(x),$$

$$a_0(x) = 1,\ a_1(x) = 2,\ a_2(x) = 4,\ a_3(x) = 8,$$

For $n > 3$, we can write the following equations:

$$a_4(x) = 2x^2 a_2(x) + 4a_0(x)$$

$$a_5(x) = 2x^2 a_3(x) + 4a_1(x)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_n(x) = 2x^2 a_{n-2}(x) + 4a_{n-4}(x)$$

Adding all these equations term by term we get

$$S_n(x) - a_0(x) - a_1(x) - a_2(x) - a_3(x)$$

$$= 2x^2(S_n(x) - a_0(x) - a_1(x) - a_{n-1}(x) - a_n(x)$$

$$+4\big(S_n(x) - a_{n-3}(x) - a_{n-2}(x) - a_{n-1}(x) - a_n(x)\big).$$

Substituting initial values we have

$$S_n(x) = \frac{(2x^2+4)[a_n(x) + a_{n-1}(x)]}{2x^2+3} + \frac{4\,[a_{n-2}(x) + a_{n-3}(x)] + 6x^2 - 15}{2x^2+3}. \quad (26)$$

Let's obtain respectively $S_{2n}(x)$ and $S_{2n+1}(x)$ using the equation (26),

$$S_{2n}(x) = \frac{(2x^2+4)[a_{2n}(x) + a_{2n-1}(x)]}{2x^2+3} + \frac{4\,[a_{2n-2}(x) + a_{2n-3}(x)] + 6x^2 - 15}{2x^2+3}. \quad (27)$$

$$S_{2n+1}(x) = \frac{(2x^2+4)[a_{2n+1}(x) + a_{2n}(x)]}{2x^2+3} + \frac{4\,[a_{2n-1}(x) + a_{2n-2}(x)] + 6x^2 - 15}{2x^2+3}. \quad (28)$$

Using (27), (28), Theorem 1 and the fact that $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ *for* $n \geq 2$ we have

$$\sum_{k=0}^{2n} a_k\,(x) = \frac{6x^2 - 15}{2x^2+3} + \frac{2^{n+1}[(2x^2+9)F_n(x^2) + (9-2x^2)F_{n-1}(x^2)]}{2x^2+3} \quad (29)$$

$$\sum_{k=0}^{2n+1} a_k\,(x) = \frac{6x^2 - 15}{2x^2+3} + \frac{2^{n+1}[(6x^2+15)F_n(x^2) + 12F_{n-1}(x^2)]}{2x^2+3} \quad (30)$$

The proof is completed.

Writing $x = 1$ in (29), we have

$$\sum_{k=0}^{2n} a_k = \frac{2^{n+1}[11F_n + 7F_{n-1}] - 9}{5}.$$

And using the definition of the Fibonacci sequence, we obtain

$$\sum_{0 \leq k \leq 2n} a_k = \frac{2^{n+1}}{5}[F_{n+3} + F_{n+5}] - \frac{9}{5},$$

which is the summation formula given in (8).

Similarly writing $x = 1$ in (30), we obtain the summation formula given in

2.4. The combinatorial representation of the polynomial sequence

The explicit formula of Fibonacci polynomial sequence is given in [4] by the formula:

$$F_n(x) = \sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-1-i}{i} x^{n-1-2i} \quad (31)$$

You can see [3] for the equivalent formula as follows

$$F_n(x) = \frac{1}{2^{n-1}} \sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n}{2i+1} x^{n-2i-1}(x^2+4)^i \quad (32)$$

We use the explicit formula (32) to prove the following corollary.

Corollary 1. *Let* $(a_n(x))$ *is the polynomial sequence defined by (10). Then for* $n \geq 1$ *we have*

$$a_{2n}(x) = 4 \sum_{i=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \left[x^2\binom{n}{2i-1} + \binom{n-1}{2i-1}\right] x^{2n-4i}\,(x^4+4)^{i-1},$$

$$a_{2n+1}(x) = 8 \sum_{i=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \left[x^2\binom{n}{2i-1} + \binom{n-1}{2i-1}\right] x^{2n-4i}\,(x^4+4)^{i-1}$$

where the limit of the index of the summations are irrelevant, as $\binom{n}{k} = 0$ *for* $k < 0$ *and* $k > n$.

Proof. Using Theorem 1 and (32) we obtain the explicit formulas easily.

2.5. Some basic properties of the polynomial sequence

We will give asymptotic behaviour of the polynomial sequence, Honsberger's formula and derivative of the polynomial sequence.

Proposition 1. (Asymptotic behaviour of the quotient of the consecutive terms)

$$\lim_{n\to\infty} \frac{a_{2n+1}(x)}{a_{2n}(x)} = 2\,, \tag{33}$$

$$\lim_{n\to\infty} \frac{a_{2n+2}(x)}{a_{2n+1}(x)} = \frac{x^2+\sqrt{x^4+4}}{2}. \tag{34}$$

Proof. (33) and (34) are immediate consequences of Theorem 1 and Binet's formula of Fibonacci polynomials given in (3).

If $x = 1$, we have

$$\lim_{n\to\infty} \frac{a_{2n+1}}{a_{2n}} = 2,$$

$$\lim_{n\to\infty} \frac{a_{2n+2}}{a_{2n+1}} = \frac{1+\sqrt{5}}{2}.$$

Proposition 2.

$$a_{2m+2n}(x) = 2^{m+n}[F_{m+1}(x^2)\big(2F_n(x^2)+F_{n-1}(x^2)\big)$$
$$+F_m(x^2)\big(2F_{n-1}(x^2)+F_{n-2}(x^2)\big)]$$

where $F_n(x)$ is the nth Fibonacci polynomial.

Proof. For m, n integers Honsberger's formula is given in [3] as follows:

$$F_{m+n}(x) = F_{m+1}(x)F_n(x) + F_m(x)F_{n-1}(x) \tag{35}$$

The identity is easily obtained from (35) and Theorem 1.

Proposition 3. Let $F_n(x)$ be the nth Fibonacci polynomial. For $n \geq 2$ derivatives of $a_{2n}(x)$ and $a_{2n+1}(x)$ are given as follows:

$$a'_{2n}(x) = \frac{x\,2^{n+1}}{x^4+4}[(2n-2)F_{n+1}(x^2)+(n-2)F_n(x^2)$$
$$+(2n+2)F_{n-1}(x^2)+nF_{n-2}(x^2)],$$

$$a'_{2n+1}(x) = \frac{x\,2^{n+2}}{x^4+4}[(2n-2)F_{n+1}(x^2)+(n-2)F_n(x^2)$$
$$+(2n+2)F_{n-1}(x^2)+nF_{n-2}(x^2)].$$

Proof. The relation between Fibonacci polynomial sequence and its derivative sequence is given in [3] as follows:

$$F'_n(x) = \frac{nF_{n+1}(x) - xF_n(x) + nF_{n-1}(x)}{x^2+4} \tag{36}$$

Derivatives of $a_{2n}(x)$ and $a_{2n+1}(x)$ are easily obtained using Theorem 1, (36) and the definition of the polynomial sequence $F_n(x)$.

3. Conclusion

We first defined a polynomial sequence $(a_n(\mathrm{x}))_{n\geq 0}$ which is an extension of the integer sequence studied in detail in [6]. We got the closed form formula of the polynomial sequence $(a_n(\mathrm{x}))_{n\geq 0}$ using the generating function method. Then we obtain some basic properties of the polynomial sequence.

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