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Research Article

Numerical Solution of Transmission Line PDEs Using Finite Difference

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Abstract

Transmission lines refer to a variety of electrical structures that transfer information or energy typically in the form of carrying electromagnetic waves. Examples of transmission lines include coaxial cables, telephone wires, microstrips, and optical fibers. Understanding the transmission and distribution of the electromagnetic waves across the line is critical for matching the load with the generator to deliver the energy or information with minimum losses. The flow of electromagnetic waves across the line is described based on the voltage and current using Partial Differential Equations (PDEs). In this paper we apply the Central Space Central Time (CSCT) finite difference numerical method to solve the transmission line PDEs. We present the numerical solution of the waveforms and compare it with the analytical solution to evaluate the accuracy of this numerical method in solving the transmission line problem. It is found that the numerical solution of the voltage waveform is very near the analytical result with small error margin. However, while the numerical solution of the current shows the same waveform as the analytical one, there is some quite significant error in the magnitude. The error is found to result from the fact that the waveform of the numerical solution has some phase shift from that of the analytical solution. Adjusting the phase shift of the current waveform results in having good agreement between numerical and analytical results.

Keywords: Transmission line, PDEs, CSCT, finite difference, numerical analysis, stability analysis, electromagnetic waveform.

Sonlu Fark Kullanarak İletim Hattı PDE'lerinin Sayısal Çözümü

Öz

İletim hatları, tipik olarak elektromanyetik dalgalar taşıma biçiminde bilgi veya enerji aktaran çeşitli elektrik yapılarını ifade eder. İletim hatlarına örnek olarak koaksiyel kablolar, telefon kabloları, mikro şeritler ve optik fiberler verilebilir. Elektromanyetik dalgaların hat boyunca iletimini ve dağıtımını anlamak, enerjiyi veya bilgiyi minimum kayıpla iletmek için yükü jeneratörle eşleştirmek için kritik öneme sahiptir. Hat boyunca elektromanyetik dalgaların akışı, Kısmi Diferansiyel Denklemler (PDE'ler) kullanılarak voltaj ve akıma dayalı olarak tanımlanır. Bu yazıda, iletim hattı PDE'lerini çözmek için Merkezi Uzay Merkezi Zaman (CSCT) sonlu farklar sayısal yöntemini uyguluyoruz. Dalga biçimlerinin sayısal çözümünü sunuyoruz ve bu sayısal yöntemin iletim hattı problemini çözmedeki doğruluğunu değerlendirmek için analitik çözümle karşılaştırıyoruz. Gerilim dalga biçiminin sayısal çözümünün, küçük hata payı ile analitik sonuca çok yakın olduğu bulunmuştur. Bununla birlikte, akımın sayısal çözümü analitik olanla aynı dalga biçimini gösterse de, büyüklükte oldukça önemli bazı hatalar vardır.

Hatanın, sayısal çözümün dalga biçiminin analitik çözümden bir miktar faz kaymasına sahip olmasından kaynaklandığı bulunmuştur. Mevcut dalga formunun faz kaymasını ayarlamak, sayısal ve analitik sonuçlar arasında iyi bir uyum sağlar.

Anahtar Kelimeler: İletim hattı, PDE'ler, CSCT, sonlu farklar, sayısal analiz, kararlılık analizi, elektromanyetik dalga formu.

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1. Introduction

The term transmission line is used to refer to a variety of structures that transfer information or energy but here the focus is on the analysis of transmission lines carrying electromagnetic waves. Examples of such transmission lines include coaxial cables, telephone wires, microstrips, and optical fibers (Transmission line theory, 2009) and (Ulaby, 1994). Such transmission lines are used for carrying audio, video, and digital information to TVs, computers, phones, etc. (Korzeniewska, 2019). A transmission line is a two-port network where at one port there is a source sending some type of electromagnetic signal (generator) while at the other port a load is receiving that signal (Davoli, 2018). Understanding the transmission and distribution of the electromagnetic signal across the line is critical for matching the load with the generator to successfully deliver the energy or information with minimum losses. This understanding is also critical for the design of efficient transmission lines in terms of length, size, and materials.

The problem of electromagnetic waves inside transmission lines is typically solved in the complex domain using the notation of phasors. So, the equations describing the transmission line are typically written and solved in phasor notation and not in the usual time domain notation. After solving the problem, the solution is transformed into the time domain notation (Daafouz, 2014). So, it is interesting to approach this problem in a new way that is not explored in the textbooks on circuits and electromagnetics. It is interesting to see how this problem can be solved using numerical methods. This approach can help us see the problem from another perspective. In particular, we can get more insight about the role of boundary conditions and initial conditions in solving partial differential equations (PDEs). This is because in the usual way this problem is solved, the engineer is more concerned with applying the already obtained equations rather than deriving the solution from the fundamental PDEs. Also, we can get to compare the results of our numerical solution with the analytical solution and thus be able to judge the suitability of the numerical method with this type of problems. Simply, the problem is not approached using the typical time domain analytical approach nor the numerical approach and it is interesting to try to apply a relatively unusual numerical technique to solve this problem and compare the numerical solution with the analytical solution to investigate the accuracy of the technique and its suitability for this kind of problems.

In order to analyze the behavior of the electromagnetic wave across the line, an appropriate electric circuit model of the line must be used. As it is well-known, a typical equivalent circuit model of the transmission line has four components: resistance, inductance, conductance, and capacitance (Ulaby, 1994) and (Wang, 2018). They can provide a very accurate model of all transmission lines carrying any form of electromagnetic signals (Ulaby, 1994). R' is the resistance per unit length measured in Ω/m. L' is the inductance per unit length measured in H/m. G' is the conductance per unit length measured in S/m. C' is the capacitance per unit length measured in F/m. The values of these parameters are unique characteristics of each transmission line that are determined by the type of the material of the line and its geometry. Based on this model, the propagation of electromagnetic signals along the line can be analyzed.

2. Methodology

Equations (1) and (2) are called the transmission line equations or the telegrapher's equations (Ulaby, 1994). They are coupled linear first-order Partial Differential Equations (PDEs) of voltage and current with respect to time and spatial position. Since the equations have two derivatives with respect to position and two derivatives with respect to time, we need two boundary conditions and two initial conditions to solve the equations. These conditions are determined by the generator supply voltage, the nature of the load, specifically the impedance of the load, and the parameters of the transmission line. So, in order to solve the equations, we have to choose a specific transmission line scenario where the generator and load are known, and the values of the transmission line parameters are given.

$$-\frac{\partial v(z,t)}{\partial z} = R' * i(z,t) + L' * \frac{\partial i(z,t)}{\partial t} \quad (1)$$

$$-\frac{\partial i(z,t)}{\partial z} = G' * v(z,t) + C' * \frac{\partial v(z,t)}{\partial t} \quad (2)$$

We assume that z = 0 at the load and z = l at the generator, where 'l' is the length of the transmission line. Assume for our problem that l = 1 m. Let the generator be connected to a load with negligible impedance, e.g., short-circuited line with Z_L=0 Ω, through a transmission line with the following parameters $R' = 10$ Ω/m, L'= 1 μH/m, $G' =$ 0.01 S/m, $C' = 1\ \text{pF/m}$. Then, we can plug in these values in Equations (1) and (2) to obtain the PDEs that describe this particular transmission line problem. Equations (3) and (4) are the PDEs that we are going to solve numerically.

$$-\frac{\partial v(z,t)}{\partial z} = 10 * i(z,t) + 10^{-6} * \frac{\partial i(z,t)}{\partial t} \quad (3)$$

$$-\frac{\partial i(z,t)}{\partial z} = 0.01 * v(z,t) + 10^{-12} * \frac{\partial v(z,t)}{\partial t} \quad (4)$$

To fully define our problem, assume a typical scenario where the generator generates a sinusoidal source voltage signal given by (note: angles are in degree not radian)

$$v(z,0) = 20 * \sin(10\pi * z) \quad (5)$$

This is our first initial condition for the voltage at t = 0. The frequency of the signal is 1 GHz as can be seen from the source voltage equation. Usually, the phase velocity of electromagnetic waves in media is lower than their wave velocity in vacuum (the speed of light 'c'). So, assume that the phase velocity in this transmission line is equal to 2/3 c. It is always beneficial to calculate the wavelength of the electromagnetic wave and compare it to the length of the transmission line as it turns out to have many connections that help understand the physics of the problem.

$$\lambda = \frac{v}{f} = \frac{\frac{2}{3} * 3 * 10^8}{1 * 10^9} = 0.2\, m \tag{6}$$

So, l = 5λ. Later we will comment on the implication of this relation between the line length and the wavelength. The current boundary condition at the load, voltage boundary condition at the load, and initial current are, respectively,

$$\mathrm{i}(0,\mathrm{t}) = 3.18 * 10^{-3} * \cos(2\pi * 10^9 * \mathrm{t} - 179.91) \tag{7}$$

$$\mathrm{v}(0,\mathrm{t}) = 0 \tag{8}$$

$$\mathrm{i}(\mathrm{z},0) = 0.00159 * \cos(10\pi * \mathrm{z} - 179.91) + 0.00159 * \cos(-10\pi * \mathrm{z} - 179.91) \tag{9}$$

These equations provide the necessary boundary and initial conditions for the voltage and current that are needed to solve the PDEs. Now, with the PDEs set and initial/ boundary conditions given we can proceed with solving the PDEs numerically. Note that 0 ≤ z ≤ 1 m and 0 ≤ t ≤ 1 μs. This time is long enough compared with the frequency and speed of the wave with respect to the length of the transmission line; long enough for the system to have reached steady state and remained in it long enough.

3. Solution and Results

To solve these PDEs numerically, we apply the finite difference method. We use the central difference for both the time derivative and the space derivative (CSCT) so that the error be of second order in both 't' and 'z'. Applying the central difference on the PDEs and rearranging the terms to make them explicit (can be solved explicitly where we solve for the future value at a single node in terms of only past values), we obtain.

$$\mathrm{i}_i^{n+1} = -20 * 10^6 * \Delta t * \mathrm{i}_i^n + \mathrm{i}_i^{n-1} - 10^6 * \frac{\Delta t}{\Delta z} * \mathrm{v}_{i+1}^n + 10^6 * \frac{\Delta t}{\Delta z} * \mathrm{v}_{i-1}^n \tag{10}$$

$$\mathrm{v}_i^{n+1} = -2 * 10^{10} * \Delta t * \mathrm{v}_i^n + \mathrm{v}_i^{n-1} - 10^{12} * \frac{\Delta t}{\Delta z} * \mathrm{i}_{i+1}^n + 10^{12} * \frac{\Delta t}{\Delta z} * \mathrm{i}_{i-1}^n \tag{11}$$

In order to make sure that the PDEs can actually be solved using suitable space/ time discretizations, we have to check the stability of the numerical method for our problem. The method must be either unconditionally stable or conditionally stable. In the second case, we have to choose our discretizations based on the stability conditions so that we can ensure that we obtain a stable solution using our numerical method. Applying Von Neumann stability analysis, the solution is stable if the magnification factor (amplitude at next or current time step over amplitude at previous time step) of the wave amplitude for both the voltage and current at each time step is less than one. That is,

$$\left| \frac{\mathrm{A_i} e^{j\omega(n+1)\Delta t}}{-\frac{2\Delta t \mathrm{R}'}{\mathrm{L}'} * \mathrm{A_i} e^{j\omega n \Delta t}} \right| = \left| \frac{1}{-\frac{2\Delta t \mathrm{R}'}{\mathrm{L}'}} \right| = \frac{\mathrm{L}'}{2\Delta t \mathrm{R}'} < 1 \tag{12}$$

$$\left| \frac{\mathrm{B_i} e^{j\omega(n+1)\Delta t}}{-\frac{2\Delta t \mathrm{G}'}{\mathrm{C}'} * \mathrm{B_i} e^{j\omega n \Delta t}} \right| = \left| \frac{1}{-\frac{2\Delta t \mathrm{G}'}{\mathrm{C}'}} \right| = \frac{\mathrm{C}'}{2\Delta t \mathrm{G}'} < 1 \tag{13}$$

So, the conditions for stability are

$$\frac{\mathrm{L}'}{2\mathrm{R}'} < \Delta t \tag{14}$$

$$\frac{\mathrm{C}'}{2\mathrm{G}'} < \Delta t \tag{15}$$

Plugging in the values we have for the line parameters, $5 * 10^{-8} < \Delta t$, $5 * 10^{-11} < \Delta t$. So, we have to choose Δt to be larger than $5 * 10^{-8}$ for stability. It is noted that the solution is unconditionally stable for any space discretization. That is, the stability does not depend on Δz.

Given that λ = 0.2, our Δz has to be much less than 0.2. Generally, Δz ≪ λ, because the wavelength is the characteristic spatial length of the wave, the wave variation in space happens in fractions of λ. So, if our space discretization is equivalent to λ, then we are missing information on the waveform of the wave. That's the wave is hugely changing within our discretization step. This is obviously not acceptable and will result in failure of the numerical method since the wave would be much varying during our discretization step. So, we can choose Δz = 0.002λ = 4*10^{-4}. Thus, given that the length of the transmission line is l = 5λ, we have 2500 nodes. For the time discretization, $\Delta t = 6.25 * 10^{-8}$. So, we have 16 time steps in the interval $0 \leq t \leq 1\ \mu s$. To do the actual computations, we make use of MATLAB. We have to develop a MATLAB program that solves the equations for each time step and saves the solution.

The problem can be analytically solved in more than one way, here we used the usual electric circuit solution that makes use of the complex domain and phasors. The main step in the solution is to obtain the impedance of the circuit based on the model and the values of the parameters given before. Then after completing the solution and transforming the results from the complex domain to the real domain, the solution for the voltage and current was obtained as shown in Equations (16) and (17).

$$v(z,t) = 10\,[\,\sin(2\pi * 10^9 * t + 10\pi * z) - \sin(2\pi * 10^9 * t - 10\pi * z)\,] \quad (16)$$

$$i(z,t) = 0.00159\,[\,\cos(2\pi * 10^9 * t + 10\pi * z - 179.91) + \cos(2\pi * 10^9 * t - 10\pi * z - 179.91)\,] \quad (17)$$

Figure 1. Surface plot of the voltage waveform along the transmission line up to t= 10^-7.

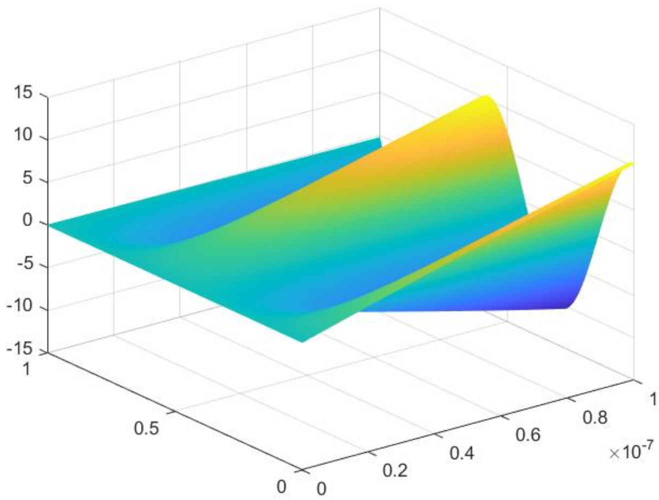


Figure 2. Surface plot of the current waveform.

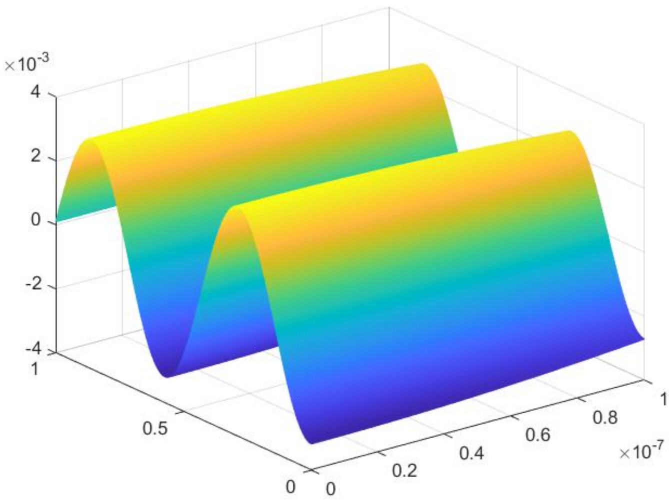


Figure 3. Voltage waveform analytical result and numerical result along the transmission line at t= 10^-7 s.

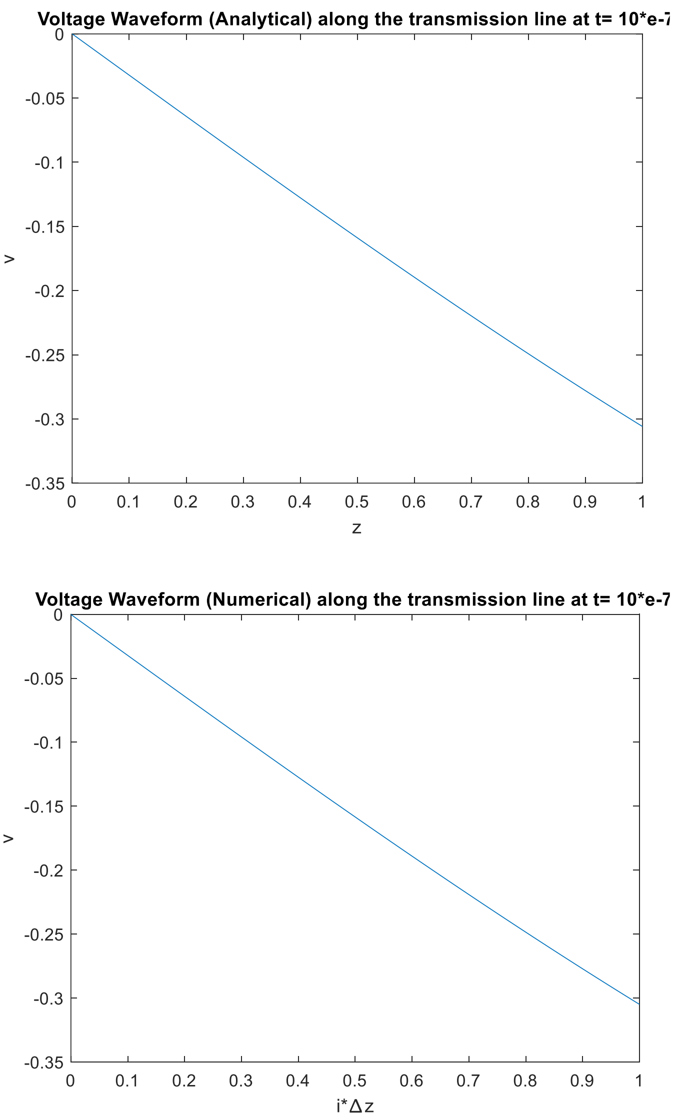
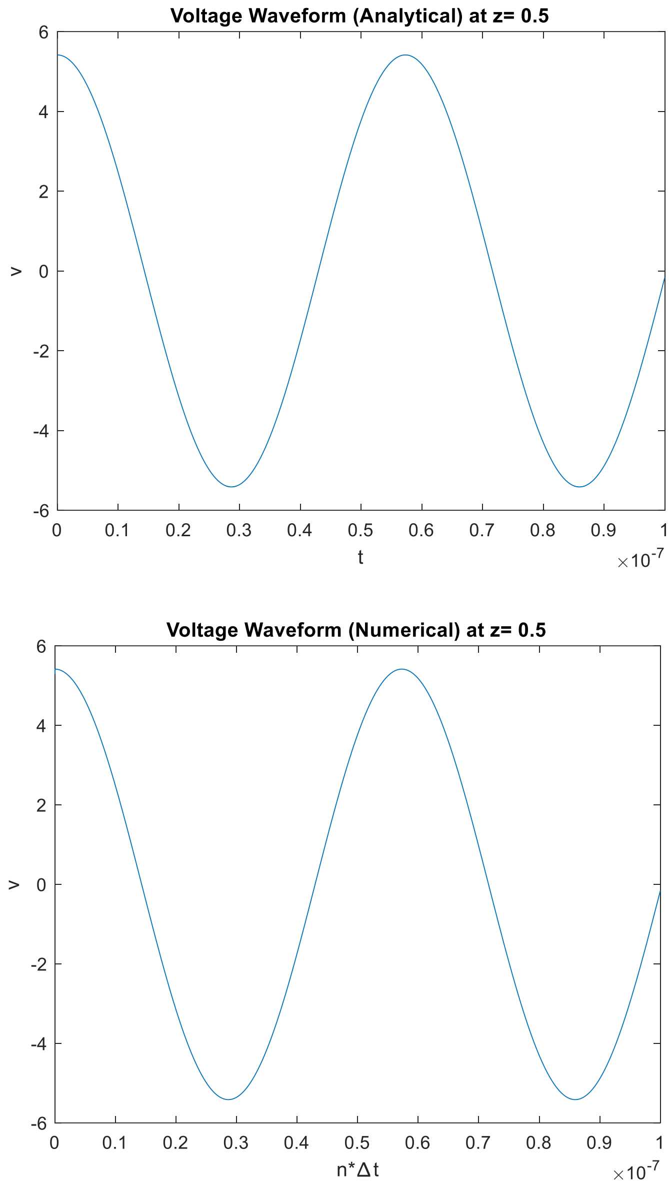
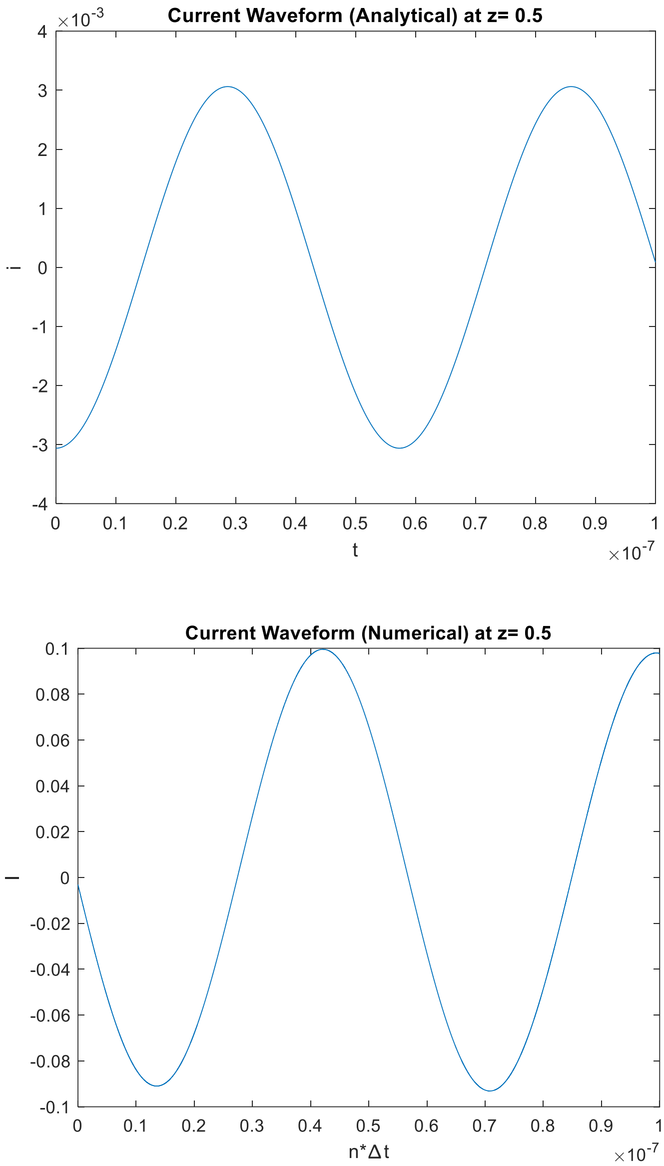


Figure 4. Voltage waveform analytical result and numerical result at the middle of the transmission line.





4. Conclusion

It can be seen that the numerical solution of the voltage is very near from the analytical result with small error that is hardly visible from the graphs. However, while the numerical solution of the current showed the same waveform as the analytical one, there was quite significant error between the magnitudes of both results. It appears that the waveform of the numerical solution has some phase shift from that of the analytical solution. So, when we compare them we get different magnitudes since we are not accounting for this potential phase shift that may have arose from accumulation of the errors. Adjusting the phase shift of the current waveform results in having good agreement between numerical and analytical results. One challenge is that the execution of the numerical code (the “for” loops in particular) takes a lot of time.

Figure 5. Current waveform analytical result and numerical result at the middle of the transmission line.

This is because the computations are very extensive since we are talking about roughly 250 million iterations. In fact, if we try to increase the time duration we are solving for by 10 times, Matlab shows an overflow error message that it cannot store the resulting variable. We had to use “single” precision instead of “double” precision for our floating-point variables so that the code be executed withing an acceptable time limit. However, this will come at the expense of accuracy at some point.

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